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Stationary response of nonlinear magneto-piezoelectric energy harvester systems under stochastic excitation

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Abstract. Recent years have shown increasing interest of researchers in energy harvesting systems designed to generate electrical energy from ambient energy sources, such as mechanical excitations. In a lot of cases excitation patterns of such systems exhibit random rather than deterministic behaviour with broad-band frequency spectra. In this paper, we study the efficiency of vibration energy harvesting systems with stochastic ambient excitations by solving corresponding Fokker-Planck equations. In the system under consideration, mechanical energy is transformed by a piezoelectric transducer in the presence of mechanical potential functions which are governed by magnetic fields applied to the device. Depending on the magnet positions and orientations the vibrating piezo beam system is subject to characteristic potential functions, including single and double well shapes. Considering random excitation, the probability density function (pdf) of the state variables can be calculated by solving the corresponding Fokker-Planck equation. For this purpose, the pdf is expanded into orthogonal polynomials specially adapted to the problem and the residual is minimized by a Galerkin procedure. The power output has been estimated as a function of basic potential function parameters determining the characteristic pdf shape.

1 Introduction

The usage of mechanical energy harvesting systems (EH-sytems) exploiting ambient vibrations has become a plausible solution for powering small devices and small amount energy storage following the frequency broad band concept [1–4]. This approach is motivated by variable or even random ambient conditions of excitation and consequently a fairly wide range of excitation frequencies. Nonlinear systems are characterized by declined frequency-amplitude characteristics which naturally cover larger ranges of frequencies. Different ways of including nonlinear elements have been discussed [4] including the application of additional magnets [5,6], gravity effects [7] and stoppers with vibration amplitude limitation [8,9], as well as impact configurations [10]. The key idea lies in the exploitation of nonlinear system dynamics

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in order to improve the efficiency of energy harvesting systems in a wider range of frequency. Simultaneously, energy harvesting from random energy sources has been investigated. Stochastically excited energy harvesting systems, firstly proposed by Cottone et al. [11], were intensively studied in a number of recent papers including Monte Carlo simulations [12–14] and experimentally based approaches [15,16]. All these investigations focussed on systems with double well potentials with special interest in the dynamical response with increasing amplitudes caused by the noise-induced escape phenomenon. The computation of probability density functions by solution of corresponding Fokker-Planck equations [17] has been proposed [18,19]. However, the considered energy harvesting systems were restricted with respect to the investigated potential functions [18] or alternatively to the electrical circuit models [19]. The present paper attempts to overcome these deficiencies by application of a method of solving multi-variable Fokker-Planck equations developed in the past 15 years [20–23], which has already been used to compute first results in the context of energy harvesting systems [24]. The method is applied to problems considered in previous works using Monte Carlo simulations [12,14]. In our opinion, the proposed method has a large potential for the very fast investigation of randomly excited energy harvesting problems and could allow for the usage of optimization procedures. As a first step in this direction, a systematic investigation of the influence of the potential shape (to be influenced by the position of the magnets) on the power output is carried out.

2 Magneto-piezoelectric energy harvester model

We consider the dynamics of a magneto-piezoelectric energy harvester as shown in Fig. 1 (see [5]).

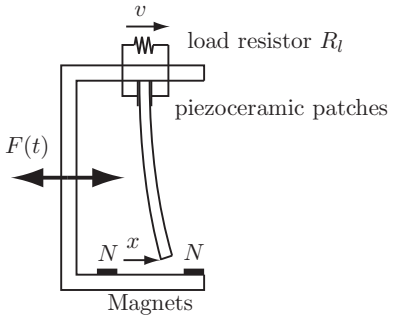


Fig. 1. Magneto-piezoelectric energy harvester [12]

The lumped-parameter system dynamics are modelled by the equations of motion

$$\ddot{x} + \gamma\dot{x} + U'(x) - \chi v = F(t) \tag{1}$$

$$\dot{v} + \lambda v + \kappa\dot{x} = 0, \tag{2}$$

where x denotes the dimsensionless displacement of the beam tip and v denotes the dimensionless voltage across the load resistor. $U'(x) = \frac{\partial}{\partial x}U(x)$ describes a conservative force field defined by an arbitrary (stable) potential function $U(x)$ in the displacement. γ denotes dimensionless linear damping, χ the dimensionless piezoelectric coupling term in the mechanical equation, κ the dimensionless piezoelectric coupling

term in the electrical equation, $\lambda \propto 1/R_l C_p$ the reciprocal of the dimensionless time constant of the electrical circuit, R_l the load resistance and C_p the capacitance of the piezoelectric material. The force $F(t)$ is proportional to the base acceleration of the device and is modelled by Gaussian white noise ξ_t, with mean and variance equal to 0 and 1, respectively. The intensity of the force $F(t)$ is characterized by σ_F, namely

$$F_t = \sigma_F \xi_t. \tag{3}$$

After introducing the notation: $X_{1,t} = x$, $X_{2,t} = \dot{x}$, and $X_{3,t} = v$, the system of equations (1-3) can be written as a vector stochastic Itô equation

$$\mathrm{d}\mathbf{X}_t = \mathbf{f}(t, \mathbf{X}_t)\mathrm{d}t + \mathbf{G}(t, \mathbf{X}_t)\mathrm{d}\mathbf{W}, \tag{4}$$

where

$$\mathbf{X}_t := \begin{pmatrix} X_{1,t} \\ X_{2,t} \\ X_{3,t} \end{pmatrix} \tag{5}$$

and

$$\mathbf{f}(t, \mathbf{X}_t) := \begin{pmatrix} X_{2,t} \\ -U'(X_{1,t}) - \gamma X_{2,t} + \chi X_{3,t} \\ -\kappa X_{2,t} - \lambda X_{3,t} \end{pmatrix}, \tag{6}$$

$$\mathbf{G}(t, \mathbf{X}_t) := \begin{pmatrix} 0 \\ \sigma_F \\ 0 \end{pmatrix}. \tag{7}$$

Capital letters with subscripts t (e.g. $X_{1,t}$) indicate stochastic processes with respect to time.

3 Weak formulation of the Fokker-Planck-equation

The statistical behaviour of a stochastic process (Eq. 4) can be described by its probability density function (pdf) $p(\mathbf{x}, t)$, which is a solution of the corresponding Fokker-Planck-equation (FPE)

$$\frac{\partial}{\partial t} p(\mathbf{x}, t) = -\sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left\{ p(\mathbf{x}, t) f_i(\mathbf{x}) \right\} + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} \left\{ p(\mathbf{x}, t) B_{ij}(\mathbf{x}) \right\}, \tag{8}$$

with $\mathbf{B}(\mathbf{x}) = \mathbf{G}\mathbf{G}^T$.

In the context of EH-systems as described above we are primarily interested in the long-term system behaviour and thus stationary pdfs $p(\mathbf{x})$. These functions are solutions of the stationary FPE, i.e. (Eq. 8) with $\frac{\partial}{\partial t} p(\mathbf{x}, t) = 0$. In our case, $\mathbf{f}(\mathbf{x})$ can be taken from (Eq. 6) and $\mathbf{B}(\mathbf{x}) = \mathbf{G}\mathbf{G}^T$ from (Eq. 7). Exact solutions of the stationary FPE are known only for very limited classes of problems, so that pdfs of general nonlinear systems are commonly computed with a variety of numerical methods. Ref. [23] demonstrates how the expansion of approximate solutions can decrease the numerical effort of solving the FPE, allowing for the computation of pdfs for comparably high-dimensional systems in a considerably shorter amount of time

usually needed by Monte-Carlo simulations. Given an approximate solution $p_0(\mathbf{x})$ of the stationary FPE, an improved solution

$$p_N(\mathbf{x}) = k_N(\mathbf{x}) p_0(\mathbf{x}) \tag{9}$$

is computed by constructing a correction function

$$k_N(\mathbf{x}) = \sum_{n \in N} c_n P_n(\mathbf{x}) \tag{10}$$

from (generally multivariate) orthogonal polynomials $P_n(\mathbf{x})$ in the state variables $x_1, x_2, \cdots$. The unknown expansion coefficients c_n are obtained in the sense of a GALERKIN-method by weak formulation of the FPE,

$$\int \cdots \int R_N(\mathbf{x}) P_j(\mathbf{x}) \mathrm{d}x_1 \cdots \mathrm{d}x_d = 0, \quad \text{for all } j \in N, \tag{11}$$

where $R_N(\mathbf{x})$ denotes the residual of the stationary FPE for $p_N(\mathbf{x})$,

$$R_N(\mathbf{x}) = -\sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left\{ p_N(\mathbf{x}) f_i(\mathbf{x}) \right\} + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} \left\{ p_N(\mathbf{x}) B_{ij}(\mathbf{x}) \right\}. \tag{12}$$

This approach has proven to be particularly efficient for high-dimensional systems where $p_0(\mathbf{x})$ is given as the product of univariate pdfs in the state variables

$$p_0(\mathbf{x}) = p_0^{(1)}(x_1) \cdots p_0^{(d)}(x_d), \tag{13}$$

leading to correction functions of the form

$$\begin{aligned} k_N(\mathbf{x}) &= \sum_{n \in N} c_n P_n(\mathbf{x}) \\ &= \sum_{n_1=0}^{N_1} \cdots \sum_{n_d=0}^{N_d} c_{n_1 \cdots n_d} P_{n_1}^{(1)}(x_1) \cdots P_{n_d}^{(d)}(x_d). \end{aligned} \tag{14}$$

4 Finding approximate solutions for EH-systems

4.1 Decoupled mechanical system

In order to obtain approximate solutions to (Eq. 1) and (Eq. 2) as the first step of the proposed method we start by looking at the mechanical part of the system, (Eq. 1), neglecting the effect of the electrical circuit (Eq. 2) on the mechanical subsystem, i.e. $\chi \approx 0$. The resulting equation of motion describes a 1-dof oscillator with linear damping under white noise excitation

$$\ddot{X}_{1,t} + \gamma \dot{X}_{1,t} + U'(X_{1,t}) = \sigma_F \xi_t. \tag{15}$$

The pdf for systems of this form (in the stationary case, $\partial p / \partial t = 0$) is found analytically and reads

$$\begin{aligned} p_{x_1 x_2}(x_1, x_2) &= p_{x_1}(x_1) p_{x_2}(x_2) \\ &= c_1 \exp\left(-\frac{2\gamma}{{\sigma_F}^2} U(x_1) \right) \frac{1}{\sigma_2 \sqrt{2\pi}} \exp\left(-\frac{1}{2{\sigma_2}^2} x_2^2 \right), \end{aligned} \tag{16}$$

where $x_1 := x, x_2 := \dot{x}$, like above. $p_{x_2}(x_2)$ is a normal distribution with mean $\mu_2 = 0$ and variance ${\sigma_2}^2 = \frac{{\sigma_F}^2}{2\gamma}$, c_1 is a normalization constant such that

$$\int_{-\infty}^{\infty} p_{x_1}(x_1)\mathrm{d}x_1 = 1, \tag{17}$$

which is computed numerically for specific $U(x)$. Apart from the fact that, according to (Eq. 16), x_1 and x_2 are independent for such systems it is worthwhile to note that the (GAUSSian) pdf of the velocity, $p_{x_2}(x_2)$, is not affected by the shape of the potential function, $U(x)$, but fully determined by the linear damping coefficient γ and the excitation intensity σ_F, which is in accordance to [19] for white noise excitation.

4.2 Electrical circuit dynamics

The problem in finding a decent estimate for the voltage pdf $p_{x_3}(x_3)$ lies in the fact that, even for $\chi = 0$ in Eq. 1 (no recoupling of the electrical dynamics on the mechanical system), $p_{x_3}(x_3)$ is affected by the potential function $U(x)$ (as opposed to $p_{x_2}(x_2)$), although the displacement x_1 does not appear in the electrical circuit equation (Eq. 2).

This is easily verified for the case of linear restoring forces, i.e.

$$U_{\mathrm{lin}}(x) = \frac{1}{2}\alpha x^2, \quad \alpha > 0, \tag{18}$$

where the system response is GAUSSian with zero mean and covariance matrix $\mathbf{K}$. The elements of the covariance matrix represent the second order moments of the random vector $\mathbf{X}$, i.e.

$$\mathbf{K}_{ij} = \langle X_i X_j \rangle, \tag{19}$$

where $\langle \cdot \rangle$ denotes the expected value of some random variable. $\mathbf{K}$ is found as a solution of the matrix equation [25]

$$\mathbf{AK} + (\mathbf{AK})^T = -\mathbf{GG}^T, \tag{20}$$

where $\mathbf{A}$ represents the linear system dynamics

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ -\alpha & -\gamma & \chi \\ 0 & -\kappa & -\lambda \end{pmatrix} \tag{21}$$

and $\mathbf{G}$ is as in Eq. 7. Note that $\mathbf{K}$ for the examined system (Eqs. 1-2) is not a diagonal matrix, i.e. the system states are correlated.

The voltage variance $\sigma_3^2 := \langle X_{3,t}^2 \rangle = \mathbf{K}_{33}$ can be evaluated analytically and yields

$$\sigma_3^2 = \frac{\kappa^2 {\sigma_F}^2}{2\left(\alpha\gamma + (\gamma + \lambda)\left(\chi\kappa + \gamma\lambda\right)\right)}, \tag{22}$$

which is a function of the restoring force parameter α in the potential function (Eq. 18). We will restrict $U(x)$ to functions of the form

$$U(x) = \frac{1}{2}\alpha x^2 + \frac{1}{4}\beta x^4, \quad \beta > 0, \tag{23}$$

so that $U(x)$ is monostable for $\alpha \geq 0$ ($x_{01} = 0$) and bistable for $\alpha < 0$ ($x_{02} = \pm\sqrt{-\frac{\alpha}{\beta}}$). Note that for potential functions as in Eq. 23, the mechanical oscillator (Eq. 15) describes a DUFFING-oscillator.

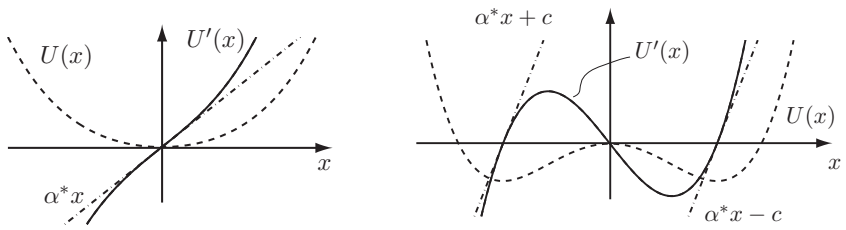
Even for the fairly narrow restriction we impose on $U(x)$ we are no longer in a position to analytically compute the voltage variance, even for $\chi = 0$. In order to obtain estimates for $p_{x_3}(x_3)$ we will instead consider the dynamics of locally linearized systems around the single stable equilibrium position $x_{01} = 0$ ($\alpha \geq 0$) or the two stable equilibrium positions $\pm x_{02}$ ($\alpha < 0$). Instead of $U_{\text{lin}}(x)$ from Eq. 18 we set

$$U^*_{\text{lin}}(x) = \frac{1}{2}\alpha^* x^2, \tag{24}$$

where

$$\alpha^* = \frac{\partial}{\partial x}U'(x)\bigg|_{x=x_{0i}} = \begin{cases} \alpha, & \alpha \geq 0 \\ -2\alpha, & \alpha < 0, \end{cases} \tag{25}$$

yielding a GAUSSian voltage pdf estimate with variance σ_3^{*2} according to Eq. 22. This estimate will yield good results for $\alpha \gg 0$ (the system oscillates around the stable equilibrium $x_{01} = 0$ with dominating linear behaviour) as well as for $\alpha \ll 0$ (the system oscillates around either one of the two stable equilibria $\pm x_{02}$ with dominating linearized behaviour, only rarely switching between the two potential wells). Note however that, even for $\alpha \approx 0$, (Eq. 22) yields a finite estimate for the voltage variance σ_3^{*2}, so that the proposed approach is applicable for arbitrary α.



(a) Monostable system ($\alpha > 0$) (b) Bistable system ($\alpha < 0$)

Fig. 2. Schematic plots of monostable and bistable potential functions and local linearization for $\alpha > 0$ (a) and $\alpha < 0$ (b).

Figure 2 shows the shape of the potential functions for monostable (Fig. 2(a)) and bistable Fig. 2(b) systems, as well as the resulting linearized restoring force laws according to Eq. 25.

In addition to Eq. 16 we thus obtain a GAUSSian approximate pdf for the voltage (here $v = x_3$),

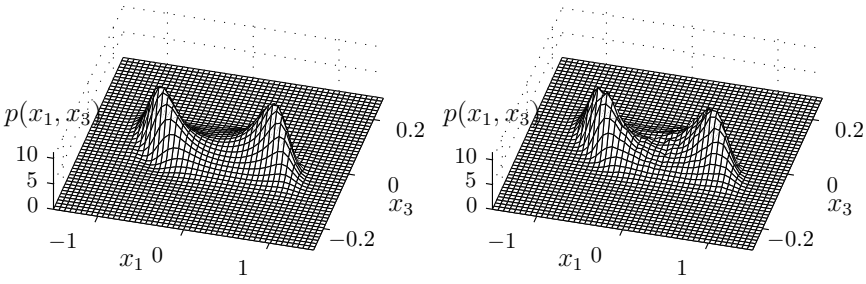
$$p_{x_3}(x_3) = \frac{1}{\sigma_3^*\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma_3^{*2}}x_3^2\right), \tag{26}$$

so that the system's approximate joint pdf is written as a decoupled pdf of the form

$$p_{x_1x_2x_3}(x_1, x_2, x_3) = p_{x_1}(x_1)p_{x_2}(x_2)p_{x_3}(x_3). \tag{27}$$

According to Eq. 9 we set

$$p_0(x) = p_{x_10}(x_1)p_{x_20}(x_2)p_{x_30}(x_3) \tag{28}$$



(a) FPE-solution ($N_i = 6$) (b) Monte-Carlo-simulation

Fig. 3. Two-dimensional marginal pdfs for bistable system with $\alpha = -0.4$, $\beta = 1$. The remaining system parameters in Eqs. (1-3) read $\sigma_F = 0.4$, $\gamma = 2.0$, $\beta = 1.0$, $\chi = 0.2$, $\lambda = 1.0$, $\kappa = 0.5$.

and

$$k_N(\mathbf{x}) = \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} \sum_{n_3=0}^{N_3} c_{n_1 n_2 n_3} P_{n_1}^{(1)}(x_1) P_{n_2}^{(2)}(x_2) P_{n_3}^{(3)}(x_3). \tag{29}$$

Application of the GALERKIN-method described above efficiently yields system pdfs for the considered mechanical EH-system.

Figure 3 shows the two-dimensional marginal pdf $p_{x_1 x_3}(x_1, x_3)$ in the displacement x_1 and the voltage x_3 for $\alpha = -0.4$, $\beta = 1$ as computed with the GALERKIN-method (Fig. 3(a)) and through Monte-Carlo-simulation (Fig. 3(b)).

5 Evaluation of potential function shape with respect to voltage output

EH-systems are designed to yield maximal electrical power outputs given some mechanical excitation. In the investigated model the electrical circuit dynamics are modelled by a constant electrical resistance R_l which is interpreted as the minimal model of an applied load consuming the electrical energy generated by the energy harvester. The electrical power P evaluates to

$$P = vi = \frac{1}{R_l} v^2, \tag{30}$$

(i denotes the electrical current) so that the expected value of the output power is proportional to the voltage variance ${\sigma_3}^2$,

$$\langle P \rangle \propto {\sigma_3}^2. \tag{31}$$

Accordingly, the goal is to design the EH-system in such a way that ${\sigma_3}^2$ becomes a maximum.

The voltage variance for the *linear* system (Eq. 22) suggests that α should be as small as possible, which obviously contradicts the need for α to be > 0 for the (linear) system to be stable. However, allowing for the restoring force law to be *nonlinear*, including the possibility of bistable potential functions, α may be varied in a larger range in order to increase ${\sigma_3}^2$.

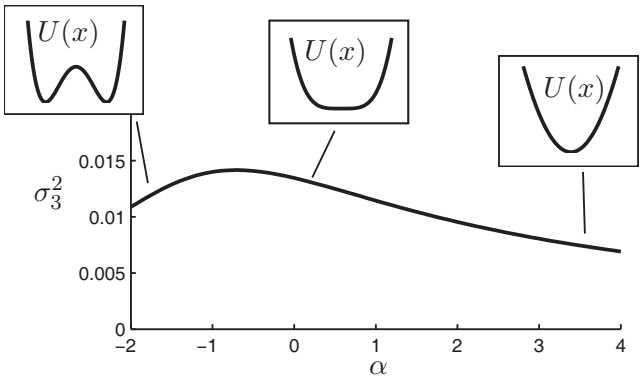


Fig. 4. Voltage output variance for varying α ($\beta = 1$, $\sigma_F = 1$), computed by solution of the FPE. Characteristic potential shapes $U(x)$ are illustrated for $\alpha = -1.5$, 0, 3.5, respectively. The remaining system parameters in Eqs. (1-3) are the same as in Fig. 3.

The Galerkin-method described above allows for the efficient computation of system pdfs and thus resulting voltage output variances. Figure 4 shows σ_3^2 for different α, as computed with the described method (based on Eqs. (16) and (26)). As can be expected from the linear system variance (Eq. 22), ${\sigma_3}^2$ decreases for $\alpha \gg 0$ (quasi-linear oscillation around $x_{01} = 0$ with large restoring term α) and $\alpha \ll 0$ (quasi-linear oscillation around either of the two equilibria $\pm x_{01}$ with large restoring term α^*).

As can be seen, there is an optimal choice of α between -1 and 0. In fact, moderately negative α, implying two stable equilibrium positions with frequent transition from one of the two potential wells to the other, leads to a maximum expected output voltage. Note that the optimal α-value further decreases for increasing excitation intensities σ_F, as higher excitation levels facilitate frequent switching between the potential wells.

6 Conclusions

In the present paper, we investigate the dynamic behavior of energy harvesting systems based on the ambient stochastic excitation of piezo beam systems. The excitation is considered to have white noise character. Due to adding magnets, the beam may have single or double potential well character. In order to describe the dynamic behavior, the corresponding Fokker-Planck equation is solved using a method introduced e.g. in [23]. This method allows the computation of pdfs in a much faster way than by using Monte Carlo methods.

The studies performed in this paper show that in the stochastic environment the double well potential is optimal for the system under consideration (Fig. 4). This is related to appearance of larger amplitude response in the presence of potential barrier overcome and cross-well oscillations. It should be noted that the term α (Eq. 21), which governs the potential shape, is determined by the choice of the magnets' positions and orientations (Fig. 1). Presumably, higher noise levels will favour higher potential barriers with stronger negative α term leading to higher voltage amplitudes and power output. However, these tendencies should be studied more systematically for various levels of excitation σ_F.

In future work, the proposed method should be applied to a more sophisticated model of the EH-system, e.g. taking into consideration more than one shape function

for the discretization of the piezo beam system under broad-band excitation, as well as a more detailed modeling of the excitation processes. The method of solution of the Fokker-Planck equation allows for a time-saving computation of the problem and therefore should enable multiple parameter variations with respect to the optimization of the energy output.

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