Chapter 6

Dynamical response of a Van der Pol system with an external harmonic excitation and fractional derivative

Arkadiusz Syta and Grzegorz Litak

Abstract We examined the Van der Pol system with external forcing and a memory possessing fractional damping term. Calculating the basins of attraction we showed broad spectrum of non-linear behaviour connected with sensitivity to the initial conditions. To quantify dynamical response of the system we propose the statistical 0-1 test. The results have been confirmed by bifurcation diagrams, phase portraits and Poincare sections.

6.1 Introduction

The system with fractional damping dependent on the velocity history have focused a lot of interest and were extensively studied in the last decade [1, 2, 3, 4, 6, 5]. To model complex energy dissipation with minimum number of parameters in presence of hysteresis and memory effect, the fractional order derivative in the damping term is proposed. In such systems the damping force is proportional to a fractional derivative of the displacement instead of the classical case (first order derivative of the displacement). The memory of the system was noted to be important factor in different areas [5, 6]. Van der Pol systems, describing relaxation-oscillations are characterized by a non-viscous composite damping term [7, 8] which is small value, negative for small amplitude oscillations and changes the sing to positive for increasing amplitude. This system property is reflected by dynamical response of limit cycle [9]. Comparing to viscous nonlinear systems this implies type of bifurcations and transition to chaos including hop bifurcations [10, 11].

Recently, Van del Pol systems have been studies in a series of papers [12, 13, 14, 15]. Pinto and Machado proposed the complex order van der Pol oscillator [12] reporting the changes in the system response spectrum with

Faculty of Mechanical Engineering, Lublin University of Technology, Nadbystrzycka 36, PL-20-618 Lublin, Poland A. Syta e-mail: a.syta@pollub.pl, G. Litak e-mail: g.litak@pollub.pl

varying the fractional order of derivative in the damping term. Attari at al. [13] focused on periodic solutions and studied system parameters for their stability. Suchorsky and Rand [14] investigated the synchroninization by a fractional coupling of two Van del Pol systems. Finally, Chen and Chen [15] studied a fractionally damped van der Pol equation with harmonic external forcing. They focus on the effect of fractional damping influence on the dynamic quasi-periodic, and chaotic responses. In particular, the transition from quasi-periodic to chaotic motion was demonstrated.

In the present paper we continue the analysis of chaotic motion proposing an efficient method for chaotic solution identification by means of the 0-1 test [18, 19]. The main idea of this method is to use the statistical asymptotics which can distinguish the periodic and non-periodic response by studying a single coordinate of system response.

6.2 Van Der Pol system with a fractional damping

The van Der Pol system with external excitation is described by equation:

$$\frac{d^2x}{dt^2} + \epsilon(x^2-1)\frac{d^qx}{dt^q} + x = f\cos(\omega t), \tag{6.1}$$

where the fractional order derivative can be described using the Grünwald - Letnikov definition [16, 17]:

$$\frac{d^qx}{dt^q} \equiv_a D_t^q x(t) = \lim_{h\to\infty} \frac{1}{h^q} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \binom{q}{j} x(t-jh), \tag{6.2}$$

where binomial coefficients can be extended to complex numbers by Euler Gamma function

$$\binom{q}{j} = \frac{q!}{j!(q-j)!} = \frac{\Gamma(q+1)}{\Gamma(j+1)\Gamma(q-j+1)}, \tag{6.3}$$

here a pair of square brackets [.] appearing in the upper limit of the sum denotes the integer part, while a the length of the memory, respectively.

Note that Eq. (6.1) can be decomposed into set of equations of lower degree:

$$\begin{aligned} {}_LD_t^1 x(t) &= y(t) \\ {}_LD_t^q x(t) &= w(t) \\ {}_LD_t^1 y(t) &= -x(t) - \epsilon(x^2(t)-1)w(t) + f\cos(\omega t), \end{aligned} \tag{6.4}$$

where w is defined as a fractional time derivative of displacement, while y coincides with velocity ($y = \dot{x}$).

6.3 Test 0-1

To quantify obtained results which can be expressed in the time series of each coordinate we use the 0-1 test for chaos detection ([18, 19, 20, 23, 24]). This test combines both spectral and statistical properties of the system and can distinguish different types of dynamic of the system by value $K \in \{0,1\}$. Below, one can find description of the method.

First of all, we change the coordinates from $(x, \dot{x})$ to the new set (p, q) defined as follows

$$p(n) = \sum_{j=1}^{n} \tilde{x}_j \cos(jc), \quad q(n) = \sum_{j=1}^{n} \tilde{x}_j \sin(jc), \tag{6.5}$$

where $\tilde{x} = [\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, ...]$ is a time series sampled from the original simulated series x using and one forth of excitation period [25]. The time interval $T/4$ $(T = 2\pi/\omega)$ corresponds to the nodal autocorrelation function of excitation harmonic term $\delta cos(\omega t)$. Note that, relevant sampling can make shorter the length of time series used in calculations leading consequently to reduction of computation time. Finally, c is a constant, $c \in (0, \pi)$. One can see that Eq. (6.5) resembles the Fourier transform for chosen frequency (in the limit of larger n).

In the next step, one computes the mean square displacement (MSD) of p and q:

$$\begin{aligned} MSD(c,j) = & \frac{1}{n-j} \sum_{i=1}^{n-j} \big\{ [p(i+j) - p(i)]^2 \\ & + [q(i+j) - q(i)]^2 \big\}, \end{aligned} \tag{6.6}$$

where $0 \ll j \ll n$ (in practice $n/100 \le j \le n/10$). The main criterion which is based on the trends of $MSD(c,j)$ in higher j limit. It is bounded for regular dynamics or unbounded for chaotic dynamics[18, 19, 20, 24, 21, 22]

The final quantity K is calculated as a asymptotic growth rate of MSD (here given by the correlation method):

$$K(c) = \frac{\text{Cov}[j, MSD(c,j)]}{\sqrt{\text{Cov}[j,j] \cdot \text{Cov}[MSD(c,j), MSD(c,j)]}}, \tag{6.7}$$

where j is based on series of natural numbers: $j = n/100, n/100 + 1..., n/10$, and Cov[$x_1$,$x_2$] denotes corresponding covariance of two series which for the same arguments $x_1 = x_2$ means variance while for chosen pair of two different series: $x_1 = j$ and $x_2 = MSD(c,j)$, it can be expressed in terms of the expectation value E[.]:

$$\mathrm{Cov}[j, MSD(c,j)] = \\ \mathrm{E}[[j - \mathrm{E}[j]] \cdot [MSD(c,j) - \mathrm{E}[MSD(c,j)]]]. \tag{6.8}$$

6.4 Simulation results

In our investigations we set $\epsilon = 8.0$, $f = 1.0$, $\omega = 3/10$, and $(x, \dot{x}) = (0.5, 0.0)$ for various q values ($q \in [0.8, 1.2]$).

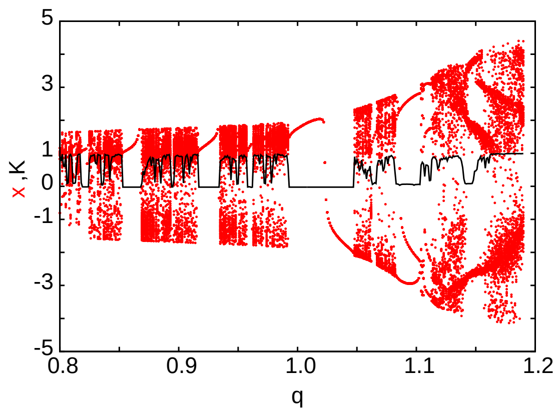

Fig. 6.1 The red points indicate the bifurcation (stroboscopic) diagram of the x coordinate versus order of the derivative $q \in [0.8, 1.2]$, initial conditions for each q were $(x, \dot{x}) = (0.5, 0.0)$. Other system parameters: $\epsilon = 8.0$, $f = 1.0$, $\omega = 3/10$. The full black line corresponds to parameter K defined for the 0-1 test versus q. Note, different q-parameter regions. $K \approx 0$ correspond to regular (periodic motion) while $K \approx 1$ to chaotic solution. The parameters used for K estimation were as follows: $n = 400$, $j = 4, .., 40$.

Figure 6.1 shows the results of the bifurcation diagram of the x coordinate s (red points) versus order of the derivative q. The characteristic broad distributions of points imply the chaotic bechaviour while the countable few points (1 to 3 points per q value noticeable in Fig. 6.1) corresponds to a periodic solution.

On the other hand the full black line corresponds to parameter K defined for the 0-1 test versus q. Note, different q-parameter regions. $K \approx 0$ correspond to regular (periodic motion) while $K \approx 1$ to chaotic solutions. Note that the $K \approx 0$ regions ideally match the broad distributions in bifurcation diagram. One can also notice some intermediate value of K (for $q = 1.05$) which could tell that reaching the asymptotic limit of K needs longer time series of $\tilde{x}$.

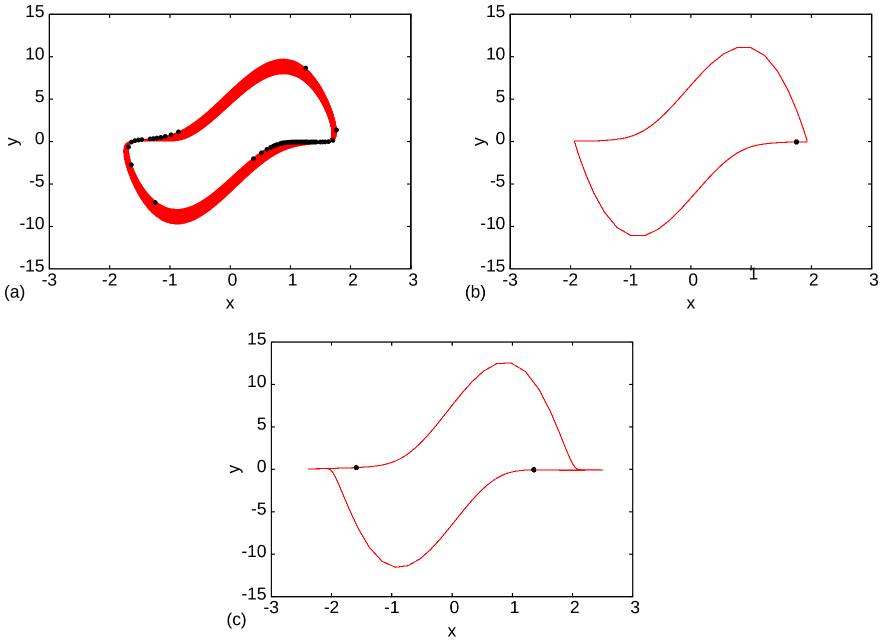

Fig. 6.2 Phase portraits and Poincare points for q =0.9 (a), q =1.0 (b), and q =1.063 (c), respectively. All other system parameters as in Fig. 6.1. The corresponding results for K: 0.91, −0.02, 0.06

For better clarity we show the phase portraits with corresponding Poincare sections in Figs. 6.2a-c. The results also confirm the 0-1 test analysis (see Fig. 6.1).

6.5 Conclusions

We have examined dynamics of the Duffing model with fractional damping term. Using nonlinear methods (phase diagrams, Poincare sections and bifurcation diagrams) we have showed significant different system response while varying the order of the derivative (from non integer to integer). We also quantified the type of motion by 0-1 test which is based on statistical properties of phase coordinate. Note that the Lyapunov exponent could be difficult to estimate as the phase space dimension is undetermined due to the memory effect. In such a situation the embedding dimension should be estimated for each q value [26].

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